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Spectral characterization of graphs whose second largest eigenvalue is less than 1 ☆

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ABSTRACT

Graphs with second largest eigenvalue $\lambda_2 \leqslant 1$ are extensively studied, however, whether they are determined by their adjacency spectra or not is less considered. In this paper we completely characterize all the connected bipartite graphs with $\lambda_2 < 1$ that are determined by their adjacency spectra. In addition, we prove that all the connected non-bipartite graphs with girth no less than 4 and $\lambda_2 < 1$ are determined by their adjacency spectra.

1. Introduction

The graphs considered in this paper are simple and undirected. Let $G = (V, E)$ be a graph with vertex set $V(G) = \{v_1, \ldots, v_n\}$ and edge set $E(G) = \{e_1, \ldots, e_m\}$. Let $A(G)$ be the $(0, 1)$-adjacency matrix of G, $P_G(\lambda) = det(\lambda I - A(G))$ the *characteristic polynomial of G with respect to $A(G)$, where I is the identity matrix*. Since $A(G)$ is real and symmetric, its eigenvalues are all real numbers, which will be ordered as $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$ and be called as the (adjacency) eigenvalues of G. The eigenvalues of G together with their multiplicities are called the *adjacency spectrum* of G. Two graphs G and H are said to be *cospectral* if they share the same spectrum (i.e., equal characteristic polynomial). A graph G is said to be determined by its adjacency spectrum (DAS for short) if for any graph H, $P_G(\lambda) = P_H(\lambda)$ implies that H is isomorphic to G. Up to now, numerous examples of cospectral but non-isomorphic graphs are

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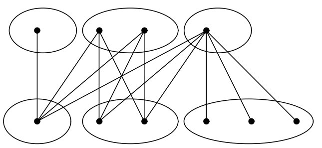

Fig. 1. $K^{(1,2,1)}_{(1,2,3)}$.

reported. But, only few graphs with very special structures have been proved to be determined by their spectra, to see [4,5,8,10,11,13,15] for references, such as the path P_n and its complement, the complete graph K_n, the cycle C_n, graph Z_n some T-shape trees, some lollipop graphs etc. Van Dam and Haemers in [4] proposed the question that which graphs are determined by their spectrum? This seems to be a difficult problem in the theory of graph spectrum.

Graphs with property $\lambda_2 \leqslant 1$ are related with graphs with least eigenvalue $\lambda_n = -2$. Some relations are given by Cvetković in [2]. Cvetković also asked if it was possible to determine all the graphs whose second largest eigenvalue λ_2 does not exceed 1. In the subsequent years, some results have been obtained, e.g., in 1989, Hong [9] determined all the trees with $\lambda_2 < 1$, and later Shu [14] determined all the trees with $\lambda_2 = 1$. Petrović [12] characterized all connected bipartite graphs with the property $\lambda_2 \leqslant 1$. In 2004, Xu [16] determined all unicyclic graphs with $\lambda_2 \leqslant 1$ and later in 2006 Xu [17] also determined all non-bipartite graphs with girth $g \geqslant 4$ and all bipartite graphs whose second largest eigenvalue is less than 1. In addition, Gao and Huang [6] determined all generalized θ-graphs whose second largest eigenvalue does not exceed 1.

In this paper we completely determine all the connected bipartite graphs with $\lambda_2 < 1$ that are DAS, and further find the cospectral mates for those that are not DAS. In particular, DAS double stars with $\lambda_2 < 1$ are obtained. As a supplement we prove that the non-bipartite graphs with girth $g \geqslant 4$ and $\lambda_2 < 1$ are all DAS.

Throughout this paper $K_{s,t}$ denotes the complete bipartite graphs with bipartition on s and t vertices, respectively, and $K_{s,t} - e$ the graph obtained from $K_{s,t}$ by deleting an edge. Denote by N and N^+ the set of non-negative integers and positive integers, respectively. The notions and symbols not defined here are standard, one can also find in [3] for references.

2. Bipartite graphs with $\lambda_2 < 1$

Here we quote some definitions in [17]. Let $(s_1, s_2, \ldots, s_r)$ and $(t_1, t_2, \ldots, t_r)$ be two ordered positive integer sets, and U_i, V_i be some vertex sets with $|U_i| = s_i$ and $|V_i| = t_i$ for $i = 1, 2, \ldots, r$. We now construct a bipartite graph $G = (U, V)$ on vertex sets $U = \cup_{i=1}^r U_i$ and $V = \cup_{i=1}^r V_i$ such that, for any given $1 \leqslant i \leqslant r$, each $u \in U_i \subset U$ is adjacent to every $v \in \cup_{1 \leqslant l \leqslant i} V_l$. For simplicity we denote such a graph $G = (U, V)$ by $K^{(s_1,\ldots,s_r)}_{(t_1,\ldots,t_r)}$. For example, $K^{(1,2,1)}_{(1,2,3)}$ is shown in Fig. 1. It is easy to see that $K^{(s_1,\ldots,s_r)}_{(t_1,\ldots,t_r)} = K^{(t_r,\ldots,t_1)}_{(s_r,\ldots,s_1)}$, $K^{(1,s)}_{(1,1)} = K_{2,s+1} - e$.

First we cite a useful theorem, in [17] due to Xu and Shao, which characterizes all connected bipartite graphs with $\lambda_2 < 1$.

Theorem 2.1 ([17]). *Assume that G is a connected bipartite graph, then $\lambda_2 < 1$ if and only if G is belonging to one of the following seven classes of graphs in Fig. 2, where a full line joining two vertex subsets represented by ellipses (or one vertex and one ellipse) indicates the edge set of the complete bipartite subgraph between these two vertex subsets, except that two vertices in these two ellipses joined by a dotted line indicates that these two vertices are not adjacent.*

(i) $\mathcal{G}_1 = \{K_{s,t} | s, t \in N^+\}$;
(ii) $\mathcal{G}_2 = \{K_{s,t} - e | s, t \in N^+\}$;
(iii) $\mathcal{G}_3 = \{K^{(1,s)}_{(1,t)} | s \in N^+, t \geqslant 2\}$;

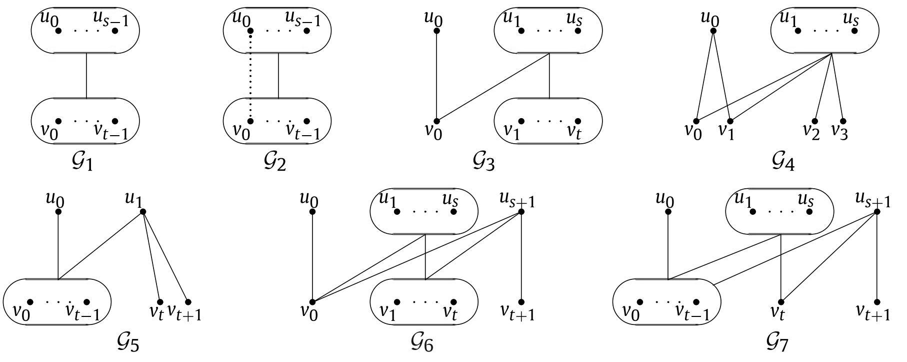

Fig. 2. $\mathcal{G}_i$ $(i = 1, \dots, 7)$.

(*iv*) $\mathcal{G}_4 = \{K_{(2,2)}^{(1,s)} | s \in N^+\}$;
(*v*) $\mathcal{G}_5 = \{K_{(t,2)}^{(1,1)} | t \geqslant 3\}$;
(*vi*) $\mathcal{G}_6 = \{K_{(1,t,1)}^{(1,s,1)} | s, t \in N^+\}$;
(*vii*) $\mathcal{G}_7 = \{K_{(t,1,1)}^{(1,s,1)} | s \in N^+, t \geqslant 2\}$.

Let G be a graph. A partition π: $V(G) = \Delta_1 \cup \Delta_2 \cup \cdots \cup \Delta_m$ is said to be *equitable* if, given $1 \leqslant i, j \leqslant m$, $c_{ij} = |N(x) \cap \Delta_j|$ is a constant for any $x \in \Delta_i$. The quotient graph (directed multigraph) with respect to π is a graph, denoted by G/π in [7], that has $\Delta_1, \Delta_2, \dots, \Delta_m$ as its vertices and has c_{ij} arcs from Δ_i to Δ_j. Therefore, the adjacency matrix of G/π is given by

$$A(G/\pi) = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1m} \\ c_{21} & c_{22} & \cdots & c_{2m} \\ & \cdots & \cdots & \\ c_{m1} & c_{m2} & \cdots & c_{mm} \end{pmatrix} \tag{1}$$

Denote by $P_G(\lambda) = |\lambda I - A|$ and $P_{G/\pi}(\lambda) = |\lambda I - A(G/\pi)|$ the characteristic polynomials of G and G/π, respectively, where I denotes the identity matrix. The following result in [7] gives a relationship of $P_G(\lambda)$ and $P_{G/\pi}(\lambda)$.

Lemma 2.2 ([7]). *If π is an equitable partition of a graph G, then the characteristic polynomial $P_{G/\pi}(\lambda)$ of the quotient graph G/π divides the characteristic polynomial $P_G(\lambda)$ of G.*

Let A_i be the adjacency matrix of graph $\mathcal{G}_i$ $(i = 1, \dots, 7)$ shown in Fig. 1. It is easy to verify that $rank(A_1) = 2$, $rank(A_j) = 4$ $(j = 2, 3, 4, 5)$ and $rank(A_k) = 6$ $(k = 6, 7)$, so $\mathcal{G}_1$, $\mathcal{G}_j$ and $\mathcal{G}_k$ has only two, four, six nonzero eigenvalues, respectively. Since $\mathcal{G}_1$ is a complete bipartite graph, it is well known that $P_{\mathcal{G}_1}(\lambda) = \lambda^{s+t-2}(\lambda^2 - st)$. Next we will give the other six graphs' characteristic polynomials in the following.

Let $V(\mathcal{G}_2) = \{u_0, u_1, \dots, u_{s-1}, v_0, v_1, \dots, v_{t-1}\}$ be the vertex set of $\mathcal{G}_2$ (see Fig. 2). Clearly, $\pi = \{\Delta_1 = \{u_0\}, \Delta_2 = \{u_1, \dots, u_{s-1}\}, \Delta_3 = \{v_0\}, \Delta_4 = \{v_1, \dots, v_{t-1}\}\}$ is a equitable partition of $V(\mathcal{G}_2)$. The adjacency matrix of the quotient graph $\mathcal{G}_2/\pi$ is

$$A(\mathcal{G}_2/\pi) = \begin{pmatrix} 0 & 0 & 0 & t-1 \\ 0 & 0 & 1 & t-1 \\ 0 & s-1 & 0 & 0 \\ 1 & s-1 & 0 & 0 \end{pmatrix}$$

and so the characteristic polynomial $P_{\mathcal{G}_2/\pi}(\lambda) = \lambda^4 - (st-1)\lambda^2 + (st + 1 - s - t)$. By Lemma 2.2 we know that $P_{\mathcal{G}_2/\pi}(\lambda)$ is a factor of $P_{\mathcal{G}_2}(\lambda)$. Moreover $\mathcal{G}_2$ has exactly four nonzero eigenvalues since $rank(A_2) = 4$. Thus

$$P_{\mathcal{G}_2}(\lambda) = \lambda^{s+t-4}(\lambda^4 - (st-1)\lambda^2 + (st+1-s-t)).$$

Similarly, we give the adjacency matrices of quotient graphs $\mathcal{G}_3/\pi, \ldots, \mathcal{G}_7/\pi$ bellow.

$$A(\mathcal{G}_3/\pi) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & t \\ 1 & s & 0 & 0 \\ 0 & s & 0 & 0 \end{pmatrix} \quad A(\mathcal{G}_4/\pi) = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 2 \\ 1 & s & 0 & 0 \\ 0 & s & 0 & 0 \end{pmatrix} \quad A(\mathcal{G}_5/\pi) = \begin{pmatrix} 0 & 0 & t & 0 \\ 0 & 0 & t & 2 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$A(\mathcal{G}_6/\pi) = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & t & 0 \\ 0 & 0 & 0 & 1 & t & 1 \\ 1 & s & 1 & 0 & 0 & 0 \\ 0 & s & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \quad A(\mathcal{G}_7/\pi) = \begin{pmatrix} 0 & 0 & 0 & t & 0 & 0 \\ 0 & 0 & 0 & t & 1 & 0 \\ 0 & 0 & 0 & t & 1 & 1 \\ 1 & s & 1 & 0 & 0 & 0 \\ 0 & s & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

By the same method, we can obtain the characteristic polynomials of graphs $\mathcal{G}_3 \ldots, \mathcal{G}_7$ and we enumerate all of them in the following lemma.

Lemma 2.3. *The characteristic polynomials of graphs $\mathcal{G}_1 - \mathcal{G}_7$ are*

(*i*) $P_{\mathcal{G}_1}(\lambda) = \lambda^{s+t-2}(\lambda^2 - st)$, $(s \geqslant 1, t \geqslant 1)$;
(*ii*) $P_{\mathcal{G}_2}(\lambda) = \lambda^{s+t-4}(\lambda^4 - (st-1)\lambda^2 + (st+1-s-t))$, $(s \geqslant 1, t \geqslant 1)$;
(*iii*) $P_{\mathcal{G}_3}(\lambda) = \lambda^{s+t-2}(\lambda^4 - (st+s+1)\lambda^2 + st)$, $(t \geqslant 2)$;
(*iv*) $P_{\mathcal{G}_4}(\lambda) = \lambda^{s+1}(\lambda^4 - 2(2s+1)\lambda^2 + 4s)$, $(s \geqslant 1)$;
(*v*) $P_{\mathcal{G}_5}(\lambda) = \lambda^{t}(\lambda^4 - 2(t+1)\lambda^2 + 2t)$, $(t \geqslant 3)$;
(*vi*) $P_{\mathcal{G}_6}(\lambda) = \lambda^{s+t-2}(\lambda^6 - (st+s+t+3)\lambda^4 + (2st+s+t+1)\lambda^2 - st)$, $(s \geqslant 1, t \geqslant 1)$;
(*vii*) $P_{\mathcal{G}_7}(\lambda) = \lambda^{s+t-2}(\lambda^6 - (st+2t+s+2)\lambda^4 + (2st+s+2t)\lambda^2 - st)$, $(s \geqslant 1, t \geqslant 2)$.

3. Spectral characterization

In this section we will determine all the connected bipartite graphs with $\lambda_2 < 1$ which are DAS, and also give the cospectral families for those graphs that are not DAS. First we have the results from Theorem 2.1 and Lemma 2.3.

Lemma 3.1. *Let G be a connected bipartite graph with $\lambda_2 < 1$ and cospectral with H. Then*

(*i*) *If $G \in \mathcal{G}_1$ then H may consist of a complete bipartite component $H_1 \in \mathcal{G}_1$ and some isolated vertices.*
(*ii*) *If $G \in \mathcal{G}_i$ $(2 \leqslant i \leqslant 5)$ then H may consist of a unique nontrivial component $H_1 \in \mathcal{G}_j$ $(2 \leqslant j \leqslant 5)$ and some isolated vertices.*
(*iii*) *If $G \in \mathcal{G}_i$ $(6 \leqslant i \leqslant 7)$ then H may consist of a unique nontrivial component $H_1 \in \mathcal{G}_j$ $(6 \leqslant j \leqslant 7)$ and some isolated vertices.*

Proof. Since H is cospectral with $G \in \mathcal{G}_1$ and G is a complete bipartite graph with only two nonzero eigenvalues and $\lambda_2(G) = 0 < 1$, we get H is also a bipartite graph with two nonzero eigenvalues and $\lambda_2(H) = 0 < 1$. Thus if H is disconnected then eigenvalue interlacing implies that only one component, say H_1, has some edges. By Theorem 2.1 and Lemma 2.3 we know that the only nontrivial component H_1 of H belongs to $\mathcal{G}_1$. Thus (i) follows.

Since H is cospectral with $G \in \mathcal{G}_i$ $(2 \leqslant i \leqslant 5)$ and, by Lemma 2.3, G has only four nonzero eigenvalues and $\lambda_2(G) < 1$, we get H is also a bipartite graph with four nonzero eigenvalues and $\lambda_2(H) < 1$. By Theorem 2.1 and Lemma 2.3 we know that the unique nontrivial component H_1 of H belongs to $\mathcal{G}_j$ $(2 \leqslant j \leqslant 5)$. Thus we obtain (ii).

Analogously, one can prove (iii). $\square$

The following result was first got by Cvetković [1] which characterizes all DAS complete bipartite graphs.

Theorem 3.2 ([1]). *A complete bipartite graph K_{s_0,t_0} is DAS if and only if the minimum of $|s-t|$ is reached for that pair (s,t) from $\{(s,t)|st=s_0t_0\}$ for which $s=s_0$ and $t=t_0$.*

By applying the Theorem 3.2, we find the following five classes of DAS complete bipartite graphs, the first two are known in [3,4], respectively.

Corollary 3.3. *From Theorem 3.2, we obtain the following*

(*i*) *If $st=p$ is a prime, then the complete bipartite graph $K_{s,t}$ is DAS* (*in [3]*).
(*ii*) *The complete bipartite graph $K_{s,s}$ is DAS* (*in [4]*).
(*iii*) *If s and t are both prime, then the complete bipartite graph $K_{s,t}$ is DAS.*
(*iv*) *The complete bipartite graphs $K_{s,s+1}, K_{s,s+2}$ are DAS.*
(*v*) *Let $p_1 \geqslant p_2 \geqslant p_3$ be primes, then the complete bipartite graph K_{p_1,p_2p_3} is DAS.*

Proof. Clearly, if $st=p$ is a prime then the set $\{|s-t||st=p\}=\{p-1\}$. Thus (i) follows by Theorem 3.2. If $s=p$ and $t=q$ are both prime then the set $\{|s-t||st=pq\}=\{|s-t|,pq-1\}$. Obviously, $|s-t|<pq-1$, and it follows (iii) by Theorem 3.2. It is a classical result that when the product of two number is a constant then the absolute value of their difference achieves its minimum if they are equal or almost equal. This implies the results (ii) and (iv). Let $p_1p_2p_3=c$, then $\{|s-t||st=c\}=\{|p_2p_3-p_1|, p_1p_3-p_2, p_1p_2-p_3, p_1p_2p_3-1\}$. Since $p_1p_2p_3-1>p_1p_2-p_3 \geqslant p_1p_3-p_2$ and

$$(p_1p_3-p_2)-(p_2p_3-p_1)=(p_1-p_2)(1+p_3)\geqslant 0,$$
$$(p_1p_3-p_2)-(p_1-p_2p_3)=(p_3-1)(p_1+p_2)\geqslant 0,$$

we have $p_1p_3-p_2 \geqslant |p_2p_3-p_1|$. Thus we obtain ($v$) by Theorem 3.2. □

Lemma 3.4. *Let $G, H \in \mathcal{G}_i$ $(2 \leqslant i \leqslant 6)$ and r be a non-negative integer, then G is not cospectral with $H \cup rK_1$.*

Proof. Let $G=K_{s_1,t_1}-e, H=K_{s_2,t_2}-e \in \mathcal{G}_2$, and assume that G and $H \cup rK_1$ are cospectral, where $s_1+t_1=s_2+t_2+r$. By Lemma 2.3 we know that

$$P_G(\lambda)=\lambda^{s_1+t_1-4}(\lambda^4-(s_1t_1-1)\lambda^2+(s_1t_1+1-s_1-t_1)),$$
$$P_{H\cup rK_1}(\lambda)=\lambda^{s_2+t_2-4+r}(\lambda^4-(s_2t_2-1)\lambda^2+(s_2t_2+1-s_2-t_2)).$$

Since G and $H \cup rK_1$ have the same characteristic polynomials, we have

$$\begin{cases} s_1+t_1-4=s_2+t_2-4+r \\ s_1t_1-1=s_2t_2-1 \\ s_1t_1+1-s_1-t_1=s_2t_2+1-s_2-t_2 \end{cases} \tag{2}$$

The latter two equations yield $s_1=t_1, s_2=t_2$ or $s_1=t_2, s_2=t_1$, and then $r=0$. Hence $G=K_{s_1,t_1}-e=K_{s_2,t_2}-e=H$.

For $\mathcal{G}_3$, $\mathcal{G}_4$, $\mathcal{G}_5$ and $\mathcal{G}_6$, the proofs are similar. □

Lemma 3.5. *Let $K_{s_1,t_1}-e \in \mathcal{G}_2$ where $\min\{s_1,t_1\}\geqslant 3$, and $K_{(1,t_2)}^{(1,s_2)} \in \mathcal{G}_3$. We have the following*

(*i*) *$K_{s_1,t_1}-e$ is not cospectral with $K_{(1,t_2)}^{(1,s_2)}$.*
(*ii*) *$(K_{s_1,t_1}-e)\cup rK_1$ is cospectral with $K_{(1,t_2)}^{(1,s_2)} \in \mathcal{G}_3$ if and only if $\frac{s_1t_1+1-s_1-t_1}{s_1+t_1-3}=t_2$ is an integer no less than 2, $s_2=s_1+t_1-3$ and $r=(s_2+t_2+2)-(s_1+t_1)$.*

(iii) $K_{(1,t_2)}^{(1,s_2)} \in \mathcal{G}_3$ *is cospectral with* $(K_{s_1,t_1} - e) \cup rK_1$ *if and only if* $\sqrt{(s_2+1)^2 - 4s_2t_2}$ *is an integer,* $s_1 = \frac{2+s_2t_2+s_2}{\frac{1}{2}s_2+\frac{3}{2}\pm\frac{1}{2}\sqrt{(s_2+1)^2-4s_2t_2}}$, $t_1 = \frac{1}{2}s_2 + \frac{3}{2} \pm \frac{1}{2}\sqrt{(s_2+1)^2 - 4s_2t_2}$ *and* $r = (s_2 + t_2 + 2) - (s_1 + t_1)$.

Proof. By the way of contradiction, let $K_{s_1,t_1} - e$ be cospectral with $K_{(1,t_2)}^{(1,s_2)}$, and so they have the same nonzero eigenvalues. By Lemma 2.3, we know that the nonzero eigenvalues of $K_{s_1,t_1} - e$ is determined by the equation

$$\lambda^4 - (s_1t_1 - 1)\lambda^2 + (s_1t_1 + 1 - s_1 - t_1) = 0, \tag{3}$$

and the nonzero eigenvalues of $K_{(1,t_2)}^{(1,s_2)}$ is determined by the equation

$$\lambda^4 - (s_2t_2 + s_2 + 1)\lambda^2 + s_2t_2 = 0. \tag{4}$$

Hence

$$\begin{cases} s_1t_1 - 1 = s_2t_2 + s_2 + 1 \\ s_1t_1 + 1 - s_1 - t_1 = s_2t_2 \end{cases} \tag{5}$$

From (5), by simple calculation, we have

$$\text{(a)} \quad \begin{cases} s_2 = s_1 + t_1 - 3 \\ t_2 = \frac{s_1t_1+1-s_1-t_1}{s_1+t_1-3} \end{cases} \quad \text{(b)} \quad \begin{cases} s_1 = \frac{2+s_2t_2+s_2}{\frac{1}{2}s_2+\frac{3}{2}\pm\frac{1}{2}\sqrt{(s_2+1)^2-4s_2t_2}} \\ t_1 = \frac{1}{2}s_2 + \frac{3}{2} \pm \frac{1}{2}\sqrt{(s_2+1)^2 - 4s_2t_2} \end{cases} \tag{6}$$

Let us define $d(s_1, t_1) = |V(K_{(1,t_2)}^{(1,s_2)})| - |V(K_{s_1,t_1} - e)|$. Since $\min\{s_1, t_1\} \geqslant 3$, from (a) in (6) we have

$$d(s_1, t_1) = (s_2 + t_2 + 2) - (s_1 + t_1) = -1 + \frac{s_1t_1 + 1 - s_1 - t_1}{s_1 + t_1 - 3}.$$

Since $\frac{\partial d(s_1,t_1)}{\partial s_1} = \frac{t_1^2-3t_1+2}{(s_1+t_1-3)^2} > 0$, also the same to $\frac{\partial d(s_1,t_1)}{\partial t_1} > 0$, we get $d(s_1, t_1)$ is strictly increasing with respect to s_1 and t_1, respectively. Hence $d(s_1, t_1) \geqslant d(3, 3) = \frac{1}{3} > 0$, which contradicts cospectral graphs having the same number of vertices. Thus (i) is obtained.

Suppose that $\frac{s_1t_1+1-s_1-t_1}{s_1+t_1-3} = t_2 \geqslant 2$ is an integer, $s_2 = s_1 + t_1 - 3$ and $r = (s_2 + t_2 + 2) - (s_1 + t_1)$. Then (a) in (6) holds. Consequently we get (5), which leads to (3) and (4). Thus we know that $(K_{s_1,t_1} - e) \cup ((s_2 + t_2 + 2) - (s_1 + t_1))K_1$ and $K_{(1,t_2)}^{(1,s_2)}$ have the same number of vertices and the same nonzero eigenvalues, and so have the same number of zero eigenvalues. Thus $(K_{s_1,t_1} - e) \cup rK_1$ and $K_{(1,t_2)}^{(1,s_2)}$ are cospectral. Conversely, let $(K_{s_1,t_1} - e) \cup rK_1$ be cospectral with $K_{(1,t_2)}^{(1,s_2)} \in \mathcal{G}_3$ where $t_2 \geqslant 2$. Then $(K_{s_1,t_1} - e)$ and $K_{(1,t_2)}^{(1,s_2)}$ share the same nonzero eigenvalues, and so (3) and (4) hold. Consequently, we obtain (5) which leads to (a) in (6), that is, $t_2 = \frac{s_1t_1+1-s_1-t_1}{s_1+t_1-3}$ is an integer, $s_2 = s_1 + t_1 - 3$ and $r = (s_2 + t_2 + 2) - (s_1 + t_1)$. Thus (ii) follows.

At last we prove (iii). Assume that $\sqrt{(s_2 + 1)^2 - 4s_2t_2}$ is an integer, r, s_1 and t_1 are defined as in (iii). If s_2 is an odd number, then $\sqrt{(s_2 + 1)^2 - 4s_2t_2}$ is an even number, from (b) in (6) we know that $t_1 = \frac{1}{2}s_2 + \frac{3}{2} \pm \frac{1}{2}\sqrt{(s_2 + 1)^2 - 4s_2t_2}$ is an integer, from (a) in (6) we note that $s_2 = s_1 + t_1 - 3$, thus s_1 is an integer too. Similarly, if s_2 is an even number we can deduce that s_1 and t_1 are both integers. Then (b) in (6) holds. Consequently we obtain (5) which leads to (3) and (4). Thus we know that $(K_{s_1,t_1} - e) \cup ((s_2 + t_2 + 2) - (s_1 + t_1))K_1$ and $K_{(1,t_2)}^{(1,s_2)}$ have the same number of vertices and the same nonzero eigenvalues, and so have the same number of zero eigenvalues. Thus $(K_{s_1,t_1} - e) \cup rK_1$ and $K_{(1,t_2)}^{(1,s_2)}$ are cospectral. Conversely, $(K_{s_1,t_1} - e)$ and $K_{(1,t_2)}^{(1,s_2)}$ share the same nonzero eigenvalues, and so (3) and (4) hold. Consequently, we obtain (5) which leads to (b) in (6), that is, $s_1 = \frac{2+s_2t_2+s_2}{\frac{1}{2}s_2+\frac{3}{2}\pm\frac{1}{2}\sqrt{(s_2+1)^2-4s_2t_2}}$

Table 1
Cospectral pairs of $(K_{s_1,t_1} - e) \cup d(s_1,t_1)K_1$ and $K^{(1,s_2)}_{(1,t_2)}$.

(s_1,t_1)	(4, 5)	(5, 10)	(6, 7)	(6, 17)	(7, 11)	(8, 9)	(8, 16)
(s_2,t_2)	(6, 2)	(12, 3)	(10, 3)	(20, 4)	(15, 4)	(14, 4)	(21, 5)
$d(s_1,t_1)$	1	2	2	3	3	3	4
(s_1,t_1)	(10, 11)	(10, 17)	(12, 13)	(14, 15)	(16, 17)	(18, 19)	
(s_2,t_2)	(18, 5)	(24, 6)	(22, 6)	(26, 7)	(30, 8)	(34, 9)	
$d(s_1,t_1)$	4	5	5	6	7	8	

and $t_1 = \frac{1}{2}s_2 + \frac{3}{2} \pm \frac{1}{2}\sqrt{(s_2+1)^2 - 4s_2t_2}$ are integers, which implies that $\sqrt{(s_2+1)^2 - 4s_2t_2}$ is an integer. Comparing the number of vertices in $(K_{s_1,t_1} - e)$ and $K^{(1,s_2)}_{(1,t_2)}$ we know that $r = (s_2 + t_2 + 2) - (s_1 + t_1)$. Thus (iii) follows. $\square$

Remark 1. If $s_1 = 2$, by definition we have $K_{s_1,t_1} - e = K^{(1,t_1-1)}_{(1,1)}$, and symmetrically $K_{s_1,t_1} - e = K^{(1,s_1-1)}_{(1,1)}$ if $t_1 = 2$. Thus, combing (i) of Lemma 3.5 we claim that $K_{s_1,t_1} - e$ where $\min\{s_1,t_1\} \geqslant 2$ has no cospectral mate $K^{(1,s_2)}_{(1,t_2)} \in \mathcal{G}_3$.

Remark 2. (ii) of Lemma 3.5 in fact gives a family (infinite) of cospectral graphs. For example:

$$(K_{2k,2k+1} - e) \cup (k-1)K_1 \text{ and } K^{(1,4k-2)}_{(1,k)} \quad (k = 2, 3, \ldots). \tag{7}$$

$$(K_{k,k^2-4k+5} - e) \cup (k-3)K_1 \text{ and } K^{(1,k^2-3k+2)}_{(1,k-2)} \quad (k = 4, 5, \ldots). \tag{8}$$

By using Matlab we list all cospectral pairs of $(K_{s_1,t_1} - e) \cup d(s_1,t_1)K_1$ and $K^{(1,s_2)}_{(1,t_2)}$ where $s_1 \leqslant 20$ and $t_1 \leqslant 20$ in Table 1.

Remark 3. (iii) of Lemma 3.5 is another version of (ii) of Lemma 3.5, we will use (iii) to determine which $K^{(1,s)}_{(1,t)} \in \mathcal{G}_3$ are DAS in the later passage.

Lemma 3.6. *Let $K_{s_1,t_1} - e \in \mathcal{G}_2$ and $K^{(1,s_2)}_{(2,2)} \in \mathcal{G}_4$, then $K_{s_1,t_1} - e$ is not cospectral with $K^{(1,s_2)}_{(2,2)}$.*

Proof. By the way of contradiction, assume they are cospectral. So they have the same nonzero eigenvalues. By Lemma 2.3, we know the nonzero eigenvalues of $K_{s_1,t_1} - e$ is determined by the Eq. (3) and the nonzero eigenvalues of $K^{(1,s_2)}_{(2,2)}$ is determined by the equation

$$\lambda^4 - 2(2s_2 + 1)\lambda^2 + 4s_2 = 0. \tag{9}$$

Hence

$$\begin{cases} s_1t_1 - 1 = 2(2s_2 + 1) \\ s_1t_1 + 1 - s_1 - t_1 = 4s_2 \end{cases} \tag{10}$$

By simple calculation, from (10) we get that $s_1 + t_1 = 4$. So we have $(s_1,t_1) = (1,3), (3,1), (2,2)$, which put into (10) we obtain $s_2 = 0, 0, \frac{1}{4}$, respectively. However, s_2 must be a positive integer, a contradiction. $\square$

By Lemma 2.3 (v), we know that the nonzero eigenvalues of $K^{(1,1)}_{(t_2,2)} \in \mathcal{G}_5$ are the roots of

$$\lambda^4 - 2(t_2 + 1)\lambda^2 + 2t_2 = 0. \tag{11}$$

If $K_{s_1,t_1} - e \in \mathcal{G}_2$ is cospectral with $K^{(1,1)}_{(t_2,2)}$ then combining (3) we have

$$\begin{cases} s_1t_1 - 1 = 2(t_2 + 1) \\ s_1t_1 + 1 - s_1 - t_1 = 2t_2 \end{cases} \tag{12}$$

Similarly one can verify that (12) has no solution of positive integers, and thus we get the following.

Lemma 3.7. *Suppose $K_{s_1,t_1} - e \in \mathcal{G}_2$ and $K^{(1,1)}_{(t_2,2)} \in \mathcal{G}_5$, then $K_{s_1,t_1} - e$ is not cospectral with $K^{(1,1)}_{(t_2,2)}$.*

Theorem 3.8. *Any graph $K_{s,t} - e \in \mathcal{G}_2$ is DAS where* $\min\{s, t\} \geqslant 2$.

Proof. Lemma 3.4 proves that $K_{s,t} - e$ has no cospectral mate in $\mathcal{G}_2$, and neither cospectral mate in $\mathcal{G}_3$, $\mathcal{G}_4$ and $\mathcal{G}_5$ by Lemmas 3.5–3.7. Thus we obtain our result by Lemma 3.1 (ii). $\square$

Remark 4. From Theorem 3.2 we know that not all complete bipartite graph $K_{s,t}$ are DAS, however, in view of Theorem 3.8 it is interesting that all connected $K_{s,t} - e$ are DAS.

Next we will determine the DAS graphs in $\mathcal{G}_3$ by first giving two lemmas.

Lemma 3.9. *Suppose $K^{(1,s_1)}_{(1,t_1)} \in \mathcal{G}_3$ and $K^{(1,s_2)}_{(2,2)} \in \mathcal{G}_4$, then $K^{(1,s_1)}_{(1,t_1)}$ is cospectral with a graph G that contains $K^{(1,s_2)}_{(2,2)}$ as induced subgraph if and only if $s_1 = 1$ and $t_1 = 4s_2$.*

Proof. Assume that $K^{(1,s_1)}_{(1,t_1)}$ and G are cospectral, then they share the same nonzero eigenvalues, which, by Lemma 2.3, are just the nonzero eigenvalues of $K^{(1,s_2)}_{(2,2)}$. Comparing the characteristic polynomials in Lemma 2.3 (iii) and (iv), we have

$$\begin{cases} s_1t_1 + s_1 + 1 = 2(2s_2 + 1) \\ s_1t_1 = 4s_2 \end{cases} \tag{13}$$

which yields that $s_1 = 1, t_1 = 4s_2$.

Conversely, assume that $s_1 = 1, s_2 = k$ $(k = 1, 2, \ldots)$ and $t_1 = 4s_2 = 4k$, clearly, Eq. (13) holds which leads to $K^{(1,1)}_{(1,4k)}$ and $K^{(1,k)}_{(2,2)}$ have the same nonzero eigenvalues. Thus $K^{(1,1)}_{(1,4k)}$ and $K^{(1,k)}_{(2,2)} \cup (3k - 2)K_1$ have the same nonzero eigenvalues, moreover they share the same number of vertices and so the same number of zero eigenvalues. We obtain $K^{(1,1)}_{(1,4k)}$ and $G = K^{(1,k)}_{(2,2)} \cup (3k - 2)K_1$ are cospectral. $\square$

From Lemma 3.9 we also obtain a family (infinite) of cospectral graphs only if $t_1 = 4k$ $(k \in N^+)$, for instance:

$$K^{(1,1)}_{(1,4k)} \text{ and } K^{(1,k)}_{(2,2)} \cup (3k - 2)K_1 \quad (k = 1, 2, \ldots). \tag{14}$$

Suppose $K^{(1,s_1)}_{(1,t_1)} \in \mathcal{G}_3$ and $K^{(1,1)}_{(t_2,2)} \in \mathcal{G}_5$ share the same nonzero eigenvalues. By Lemma 2.3 we have

$$\begin{cases} s_1t_1 + s_1 + 1 = 2(t_2 + 1) \\ s_1t_1 = 2t_2 \end{cases} \tag{15}$$

which gives that $s_1 = 1$ and $t_1 = 2t_2$. Conversely, by the same arguments as in Lemma 3.9 we get the following lemma.

Lemma 3.10. *Suppose $K^{(1,s_1)}_{(1,t_1)} \in \mathcal{G}_3$ and $K^{(1,1)}_{(t_2,2)} \in \mathcal{G}_5$, then $K^{(1,s_1)}_{(1,t_1)}$ is cospectral with a graph G that contains $K^{(1,1)}_{(t_2,2)}$ as induced subgraph if and only if $s_1 = 1$ and $t_1 = 2t_2$.*

From Lemma 3.10 we also obtain a family (infinite) of cospectral graphs only if $t_1 = 2(k + 1)(k \in N^+)$, for instance:

$$K^{(1,1)}_{(1,2(k+1))} \text{ and } K^{(1,1)}_{(k+1,2)} \cup kK_1 \quad (k = 1, 2, \ldots). \tag{16}$$

Theorem 3.11. *Let $s \geqslant 1$ and $t \geqslant 2$. A graph $K^{(1,s)}_{(1,t)} \in \mathcal{G}_3$ is DAS if and only if one of the following holds*

(*i*) $s = 1$ *and* $t = 2$ *or* $2k + 1 (k \in N^+)$;
(*ii*) $s \geqslant 2$ *and* $\sqrt{(s+1)^2 - 4st} \notin N$.

Proof. By Lemma 3.1, for the sufficiency we need to show that $K^{(1,s)}_{(1,t)}$ has no cospectral mate in $\mathcal{G}_2$–$\mathcal{G}_5$ under the conditions of (i) and (ii). First we note that $K^{(1,s)}_{(1,t)}$ has no cospectral mate in $\mathcal{G}_3$ according to Lemma 3.4.

Suppose $s = 1$. If $t = 2$, the Lemma 3.5 (iii) proves that $K^{(1,1)}_{(1,2)}$ has no cospectral mate in $\mathcal{G}_2$; the Lemma 3.9 proves that $K^{(1,1)}_{(1,2)}$ has no cospectral mate in $\mathcal{G}_4$; the Lemma 3.10 proves that $K^{(1,1)}_{(1,2)}$ has no cospectral mate in $\mathcal{G}_5$. If $t = 2k + 1$ then $4 \nmid t$, and so $K^{(1,s)}_{(1,t)}$ has no cospectral mate in $\mathcal{G}_4$ by Lemma 3.9. Since $2 \nmid t = 2k + 1$, $K^{(1,s)}_{(1,t)}$ has no cospectral mate in $\mathcal{G}_5$ by Lemma 3.10. Since $\sqrt{(s+1)^2 - 4st} = \sqrt{4 - (8k+4)} \notin N$, $K^{(1,s)}_{(1,t)}$ has no cospectral mate in $\mathcal{G}_2$ by Lemma 3.5 (iii). Thus we obtain (i) by Lemma 3.1 (ii).

Next suppose $s \geqslant 2$, by Lemmas 3.9 and 3.10 we know that $K^{(1,s)}_{(1,t)}$ has no cospectral mate in $\mathcal{G}_4$ and $\mathcal{G}_5$, respectively. Since $\sqrt{(s+1)^2 - 4st} \notin N$, $K^{(1,s)}_{(1,t)}$ has no cospectral mate in $\mathcal{G}_2$ by Lemma 3.5 (iii). Thus we obtain (ii) by Lemma 3.1 (ii).

For the necessity we need to verify that $K^{(1,s)}_{(1,t)}$ is not DAS if $s = 1$ and $t = 4, 6, 8, \ldots$, or $s \geqslant 2$ and $\sqrt{(s+1)^2 - 4st} \in N$. In fact, (16) gives us a family of cospectral mates of $K^{(1,s)}_{(1,t)}$ for $s = 1$ and $t = 4, 6, 8, \ldots$. (iii) of Lemma 3.5 gives us a family of cospectral mates of $K^{(1,s)}_{(1,t)}$ for $s \geqslant 2$ and $\sqrt{(s+1)^2 - 4st} \in N$.

It completes our proof. □

Note that Shu proved in [14] that a tree T with diameter three and $\lambda_2(T) < 1$ if and only if $T = K^{(1,1)}_{(1,n-3)}$ which is a double star. It immediately follows the following result.

Corollary 3.12. *The double star $K^{(1,1)}_{(1,n-3)}$ is DAS if and only if $n = 5$ or $n = 2(k+2) (k \in N)$.*

Now we will determine the DAS graphs in $\mathcal{G}_4$ by first giving the following lemma.

Lemma 3.13. *Suppose $K^{(1,s_1)}_{(2,2)} \in \mathcal{G}_4$ and $K^{(1,1)}_{(t_2,2)} \in \mathcal{G}_5$, then $K^{(1,s_1)}_{(2,2)}$ is not cospectral with $K^{(1,1)}_{(t_2,2)}$.*

Proof. Assume that $K^{(1,s_1)}_{(2,2)}$ and $K^{(1,1)}_{(t_2,2)}$ are cospectral, then they share the same nonzero eigenvalues, by Lemma 2.3 (iv), the nonzero eigenvalues of $K^{(1,s_1)}_{(2,2)}$ are the roots of

$$\lambda^4 - 2(2s_1 + 1)\lambda^2 + 4s_1 = 0, \tag{17}$$

by Lemma 2.3 (v), the nonzero eigenvalues of $K^{(1,1)}_{(t_2,2)}$ are determined by

$$\lambda^4 - 2(t_2 + 1)\lambda^2 + 2t_2 = 0, \tag{18}$$

thus

$$\begin{cases} 2(2s_1+1) = 2(t_2+1) \\ 4s_1 = 2t_2 \end{cases} \tag{19}$$

which yields that $s_1 = \frac{t_2}{2}$. If $t_2 = 2$, then $K_{(2,2)}^{(1,s_1)}$ and $K_{(t_2,2)}^{(1,1)}$ are overlap. If $t_2 > 2$, then we have $|V(K_{(2,2)}^{(1,s_1)})| = s_1 + 5 = \frac{t_2}{2} + 5 < t_2 + 4 = |V(K_{(t_2,2)}^{(1,1)})|$ contradicting cospectral graphs having the same number of vertices. $\square$

It is easy to verify that, from the proof of Lemma 3.13, we can find a familiy (infinite) of cospectral graphs bellow:

$$K_{(2,2)}^{(1,k)} \cup (k-1)K_1 \text{ and } K_{(2k,2)}^{(1,1)} \quad (k = 2, 3, \ldots). \tag{20}$$

Theorem 3.14. *Every graph $K_{(2,2)}^{(1,s)} \in \mathcal{G}_4$ is DAS.*

Proof. Lemmas 3.4 and 3.6 prove respectively that $K_{(2,2)}^{(1,s)}$ has no cospectral mate in $\mathcal{G}_4$ and $\mathcal{G}_2$. We claim that $K_{(2,2)}^{(1,s)}$ has no cospectral mate $K_{(1,t_1)}^{(1,s_1)} \in \mathcal{G}_3$. Since otherwise, $s_1 = 1$ and $t_1 = 4s$ by Lemma 3.9. However, in this situation, we have

$$\left|V\left(K_{(1,t_1)}^{(1,s_1)}\right)\right| = s_1 + t_1 + 2 = 4s + 3 > s + 5 = \left|V\left(K_{(2,2)}^{(1,s)}\right)\right|.$$

This is impossible. Finally Lemma 3.13 confirms that $K_{(2,2)}^{(1,s)}$ has no cospectral mate in $\mathcal{G}_5$. Thus we obtain our result by Lemma 3.1 (ii). $\square$

Theorem 3.15. *Let $t \geqslant 3$. A graph $K_{(t,2)}^{(1,1)} \in \mathcal{G}_5$ is DAS if and only if $t = 2k+1 (k \in N^+)$.*

Proof. Sufficiency, by Lemma 3.1 it suffices to show that $K_{(t,2)}^{(1,1)}$ has no cospectral mate in $\mathcal{G}_2$–$\mathcal{G}_5$ with the restriction of $t = 2k+1$. First we note that $K_{(t,2)}^{(1,1)}$ has no cospectral mate in $\mathcal{G}_5$ and $\mathcal{G}_2$ according to Lemmas 3.4 and 3.7, respectively. Since $t = 2k+1 \neq 2k$, $K_{(t,2)}^{(1,1)}$ has no cospectral mate in $\mathcal{G}_3$ by Lemma 3.10. At last, Lemma 3.13 confirms that $K_{(t,2)}^{(1,1)}$ has no cospectral mate in $\mathcal{G}_4$ unless $t = 2k$ (to see (20) for the details). Thus the sufficiency follows.

For the necessity we need to verify that $K_{(t,2)}^{(1,1)}$ is not DAS if $t = 4, 6, 8, \ldots$. In fact, (20) gives us the cospectral mate of $K_{(t,2)}^{(1,1)}$ in this case. $\square$

Theorem 3.16. *Let $t_2 \geqslant 2$. A graph $K_{(1,t,1)}^{(1,s,1)} \in \mathcal{G}_6$ is DAS if and only if the following equations*

$$s_1 = \frac{-1 + s_2 + 2t_2 \pm \sqrt{1 - 2s_2 - 4t_2 + s_2^2 + 4t_2^2}}{2} \tag{21}$$

$$t_1 = \frac{-1 + s_2 + 2t_2 \mp \sqrt{1 - 2s_2 - 4t_2 + s_2^2 + 4t_2^2}}{2} \tag{22}$$

have no positive integer solution.

Proof. Sufficiency, by Lemma 3.1 (iii) it suffices to show that $K_{(1,t,1)}^{(1,s,1)}$ has no cospectral mate in $\mathcal{G}_6$ and $\mathcal{G}_7$. First we note that $K_{(1,t,1)}^{(1,s,1)}$ has no cospectral mate in $\mathcal{G}_6$ according to Lemma 3.4. Next we prove $K_{(1,t,1)}^{(1,s,1)}$ has no cospectral mate in $\mathcal{G}_7$. By the way of contradiction, let $K_{(1,t_1,1)}^{(1,s_1,1)} \in \mathcal{G}_6$ and $K_{(t_2,1,1)}^{(1,s_2,1)} \in \mathcal{G}_7$ are

Table 2
Cospectral pairs of $K_{(t_1,1,1)}^{(1,s_1,1)}$ and $K_{(t_2,1,1)}^{(1,s_2,1)} \cup \Delta(V)K_1$.

(s_1,t_1)	(4, 1)	(6, 1)	(8, 1)	(10, 1)	(12, 1)	(14, 1)	(16, 1)	(18, 1)
(s_2,t_2)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)	(2, 7)	(2, 8)	(2, 9)
$\Delta(V)$	1	2	3	4	5	6	7	8
(s_1,t_1)	(20, 1)	(10, 2)	(9, 2)	(21, 3)	(16, 3)	(15, 3)	(36, 4)	(25, 4)
(s_2,t_2)	(2, 10)	(5, 4)	(6, 3)	(7, 9)	(8, 6)	(9, 5)	(9, 16)	(10, 10)
$\Delta(V)$	9	2	2	8	5	4	15	9

cospectral, then they share the same nonzero eigenvalues, comparing the characteristic polynomials of Lemma 2.3 (*vi*) and (*vii*) we have

$$\begin{cases} s_1t_1 + s_1 + t_1 + 3 = s_2t_2 + s_2 + 2t_2 + 2 \\ 2s_1t_1 + s_1 + t_1 + 1 = 2s_2t_2 + s_2 + 2t_2 \\ s_1t_1 = s_2t_2 \end{cases} \tag{23}$$

so

$$\begin{cases} s_1t_1 = s_2t_2 \\ s_1 + t_1 = s_2 + 2t_2 - 1 \end{cases} \tag{24}$$

which implies that

$$s_1 = \frac{-1 + s_2 + 2t_2 \pm \sqrt{1 - 2s_2 - 4t_2 + s_2^2 + 4t_2^2}}{2}$$
$$t_1 = \frac{-1 + s_2 + 2t_2 \mp \sqrt{1 - 2s_2 - 4t_2 + s_2^2 + 4t_2^2}}{2}$$

Since s_1, t_1, s_2, t_2 are positive integer number, we obtain an integral solution of Eqs. (21) and (22), a contradiction.

For the necessity we need to prove that $K_{(1,t,1)}^{(1,s,1)}$ is not DAS if Eqs. (21) and (22) have positive integer solutions. In fact, let $\Delta(V) = |V(K_{(t_1,1,1)}^{(1,s_1,1)})| - |V(K_{(t_2,1,1)}^{(1,s_2,1)})| = (s_1 + t_1) - (s_2 + t_2)$, then $K_{(t_1,1,1)}^{(1,s_1,1)}$ and $K_{(t_2,1,1)}^{(1,s_2,1)} \cup \Delta(V)K_1$ are cospectral. $\square$

As a byproduct, Theorem 3.16 in fact gives a family (infinite) of cospectral graphs. For example, let $s_1 = 2k, t_1 = 1$ and $s_2 = 2, t_2 = k$. We obtain a family of cospectral graphs bellow:

$$K_{(1,1,1)}^{(1,2k,1)} \text{ and } K_{(k,1,1)}^{(1,2,1)} \cup (k-1)K_1 \quad (k = 2, 3, \ldots). \tag{25}$$

Let $s_1 = 6k - 2, t_1 = k$ and $s_2 = 3k - 1, t_2 = 2k$. We obtain a family of cospectral graphs bellow:

$$K_{(1,k,1)}^{(1,6k-2,1)} \text{ and } K_{(2k,1,1)}^{(1,3k-1,1)} \cup (2k-1)K_1 \quad (k = 2, 3, \ldots). \tag{26}$$

By using Matlab we exhaust all the cospectral pairs of $K_{(t_1,1,1)}^{(1,s_1,1)}$ and $K_{(t_2,1,1)}^{(1,s_2,1)} \cup \Delta(V)K_1$ for $s_2 \leqslant 10$, $2 \leqslant t_2 \leqslant 10$, where $\Delta(V) = (s_1 + t_1) - (s_2 + t_2)$ in Table 2.

Lemma 3.17. *Let $t_1, t_2 \geqslant 2$ and $K_{(t_1,1,1)}^{(1,s_1,1)}, K_{(t_2,1,1)}^{(1,s_2,1)}$ be different graphs in $\mathcal{G}_7$. Then $K_{(t_1,1,1)}^{(1,s_1,1)}$ is cospectral with a graph G which contains $K_{(t_2,1,1)}^{(1,s_2,1)}$ as induced subgraph if and only if $s_1 = 2t_2, t_1 = \frac{1}{2}s_2$ and $\frac{1}{2}s_1 > t_1$.*

Proof. For the necessity, according to our assumption, $K_{(t_1,1,1)}^{(1,s_1,1)}$ and $K_{(t_2,1,1)}^{(1,s_2,1)}$ share the same nonzero eigenvalues. By comparing the characteristic polynomials in Lemma 2.3 (*vii*) we have

$$\begin{cases} s_1t_1 + s_1 + 2t_1 + 2 = s_2t_2 + s_2 + 2t_2 + 2 \\ 2s_1t_1 + s_1 + 2t_1 = 2s_2t_2 + s_2 + 2t_2 \\ s_1t_1 = s_2t_2 \end{cases} \tag{27}$$

which yields that $s_1 = s_2, t_1 = t_2$ or $s_1 = 2t_2, t_1 = \frac{1}{2}s_2$. Clearly, if $s_1 = s_2, t_1 = t_2$ then $K_{(t_1,1,1)}^{(1,s_1,1)} = K_{(t_2,1,1)}^{(1,s_2,1)}$. Hence $s_1 = 2t_2$ and $t_1 = \frac{1}{2}s_2$. Moreover, since $K_{(t_2,1,1)}^{(1,s_2,1)}$ is the induced subgraph of G which is cospectral with $K_{(t_1,1,1)}^{(1,s_1,1)}$, we have

$$\left|V\left(K_{(t_1,1,1)}^{(1,s_1,1)}\right)\right| = V(G) \geqslant \left|V\left(K_{(1,t_2,1)}^{(1,s_2,1)}\right)\right|,$$

thus

$$(s_1 + t_1 + 4) - (s_2 + t_2 + 4) = (s_1 + t_1 + 4) - \left(2t_1 + \frac{1}{2}s_1 + 4\right) = \frac{1}{2}s_1 - t_1 \geqslant 0, \tag{28}$$

equality holds if and only if $\frac{1}{2}s_1 = t_1$ and in this case $K_{(t_1,1,1)}^{(1,s_1,1)} = K_{(t_2,1,1)}^{(1,s_2,1)}$. Hence $\frac{1}{2}s_1 > t_1$, and necessity holds.

Conversely suppose that $s_1 = 2t_2, t_1 = \frac{1}{2}s_2$ and $\frac{1}{2}s_1 > t_1$. We take $t_2 = k \geqslant 2$ and $t_1 = r \geqslant 2$ then $s_1 = 2k$ and $s_2 = 2r$. It is easy to verify that

$$K_{(r,1,1)}^{(1,2k,1)} \text{ and } K_{(k,1,1)}^{(1,2r,1)} \cup (k-r)K_1, \tag{29}$$

are cospectal. $\square$

Theorem 3.18. *Let $t \geqslant 2$. A graph $K_{(t,1,1)}^{(1,s,1)} \in \mathcal{G}_7$ is DAS if and only if one of the following holds*

(*i*) $s = 2k + 1$ $(k \in N)$;
(*ii*) $s = 2k$ *and* $t \geqslant k$ $(k \in N^+)$.

Proof. For the sufficiency, by Lemma 3.1 (iii) it suffices to show that $K_{(t,1,1)}^{(1,s,1)}$ has no cospectral mate in $\mathcal{G}_6$ and $\mathcal{G}_7$ under the condition (i) or (ii). By the way of contradiction, firstly let $K_{(t_2,1,1)}^{(1,s_2,1)} \in \mathcal{G}_7 (t_2 \geqslant 2)$ be cospectral with $K_{(1,t_1,1)}^{(1,s_1,1)} \in \mathcal{G}_6$. Then, according to the arguments in the proof of Theorem 3.16, we get the Eq. (24), which gives

$$\begin{aligned} \left|V\left(K_{(t_1,1,1)}^{(1,s_1,1)}\right)\right| &= s_1 + t_1 + 4 \\ &= s_2 + 2t_2 + 3 \\ &= (s_2 + 2) + (t_2 + 2) + (t_2 - 1) \\ &> \left|V\left(K_{(t_2,1,1)}^{(1,s_2,1)}\right)\right| \end{aligned}$$

a contradiction. Next by Lemma 3.17 it is easy to see that $K_{(t_2,1,1)}^{(1,s_2,1)}$ has no cospectral mate in $\mathcal{G}_7$ under the condition (i) or (ii).

For the necessity, we need to verify that $K_{(t_2,1,1)}^{(1,s_2,1)}$ is not DAS if $t = 2k$ and $k > t$. In fact, (29) gives us the cospectral mate of $K_{(t_2,1,1)}^{(1,s_2,1)}$ in this case. $\square$

4. Non-bipartite graph with $\lambda_2 < 1$ and $g > 3$

In the end of this paper we show that all connected non-bipartite graph with girth $g > 3$ and $\lambda_2 < 1$ are DAS. Let $S(K_{s,t})$ be the graph obtained from $K_{s,t}$ by subdividing an edge of it.

Lemma 4.1 ([17]). *Suppose G is a connected non-bipartite graph with girth $g > 3$, then $\lambda_2 < 1$ if and only if $G = S(K_{s,t})$, where* $\min\{s, t\} \geqslant 2,\ s + t + 1 = |V(G)|$.

By the same method used in Lemma 2.3 we obtain the characteristic polynomial of $S(K_{s,t})$ as following

$$P_{S(K_{s,t})} = \lambda^{s+t-4}(\lambda^5 - (st+1)\lambda^3 + (3st - 2(s+t) + 1)\lambda - 2(st - (s+t) + 1)).$$

Theorem 4.2. *$S(K_{s,t})$ is DAS.*

Proof. Assume that H is cospectral with $S(K_{s,t})$, and then $\lambda_2(H) = \lambda_2(S(K_{s,t})) < 1$. By eigenvalue interlacing theorem we claim that H has exactly one nontrivial component, say H_1, then $H = H_1 \cup rK_1$ for some integer r. Moreover, cospectral graphs have the same number of triangles and $S(K_{s,t})$ has girth greater than 3. This implies H_1 also has girth greater than 3 and $\lambda_2(H_1) < 1$, by Lemma 4.1 we conclude that $H_1 = S(K_{s_1,t_1})$. Since $S(K_{s_1,t_1})$ and $S(K_{s,t})$ share the same nonzero eigenvalues, we get

$$\begin{cases} st + 1 = s_1t_1 + 1 \\ 3st - 2(s+t) + 1 = 3s_1t_1 - 2(s_1 + t_1) + 1 \end{cases} \tag{30}$$

which yields that $s = s_1, t = t_1$ or $s = t_1, t = s_1$ and hence $S(K_{s,t}) = S(K_{s_1,t_1}), r = 0$. It follows that $H = S(K_{s,t})$. $\square$

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